

# Bäcklund and Darboux transformations for the nonstationary Schrödinger equation\*

M. Boiti, F. Pempinelli, A. Pogrebkov<sup>†</sup> and B. Prinari  
Dipartimento di Fisica dell'Università  
Sezione INFN, 73100 Lecce, ITALY<sup>‡</sup>

February 7, 2008

## Abstract

Potentials of the nonstationary Schrödinger operator constructed by means of  $n$  recursive Bäcklund transformations are studied in detail. Corresponding Darboux transformations of the Jost solutions are introduced. We show that these solutions obey modified integral equations and present their analyticity properties. Generated transformations of the spectral data are derived.

## 1 Introduction

In this article we continue (see [1]–[5]) our investigation into the direct and inverse scattering transform for the nonstationary Schrödinger operator

$$L = i\partial_{x_2} + \partial_{x_1}^2 - u(x_1, x_2), \quad (1.1)$$

in the case in which  $u$  is a real function with “ray” type behavior. More exactly,  $u$  is supposed to be rapidly decaying in all directions on the  $x$ -plane with the exception of some finite number of directions, where it has finite and nontrivial limits, i.e.

$$u_{n,\pm}(x_1) = \lim_{x_2 \rightarrow \pm\infty} u(x_1 - 2\mu_n x_2, x_2), \quad n = 1, 2, \dots, N, \quad (1.2)$$

for  $N$  real constants  $\mu_n$ . Spectral theory of operator  $L$  with potential of this class is interesting *per se* and because it is associated [6, 7] to the Kadomtsev–Petviashvili equation [8]—in its version called KPI—

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = 3u_{x_2x_2}. \quad (1.3)$$

The mentioned extension of the spectral theory of the nonstationary Schrödinger operator would provide possibility to extend correspondingly the class of solutions of the KPI equation, including potentials with asymptotic one dimensional behavior. On the other side such spectral theory is essential for investigation by the inverse scattering

\*Work supported in part by PRIN 97 “Sintesi” and grant of RFBR #96-01-00344.

<sup>†</sup>Permanent address: Steklov Mathematical Institute, Gubkin str., 8, Moscow, 117966, GSP-1, Russia; e-mail pogreb@mi.ras.ru

<sup>‡</sup>e-mail boiti@le.infn.it

transform method of strongly localized soliton solutions and their interaction with background for the Davey–Stewartson I equation, where operator (1.1) controls behavior of the “boundaries at infinity” of an auxiliary function for this equation (see [9]–[10] for details).

Spectral theory of the operator (1.1) with ray potential is essentially more involved than the standard case of rapidly decaying potential. Indeed, in the standard case [11]–[18] one defines the Jost solution  $\Phi(x, k)$  as the solution of the nonstationary Schrödinger equation,

$$(i\partial_{x_2} + \partial_{x_1}^2 - u(x))\phi(x, k) = 0, \quad (1.4)$$

that is analytic in the complex plane of the spectral parameter  $k$ ,  $k_{\Im} \neq 0$ , and normalized at infinity by the condition that for the function

$$\chi(x, k) = e^{ikx_1 + ik^2 x_2} \Phi(x, k) \quad (1.5)$$

we have

$$\lim_{k \rightarrow \infty, k_{\Im} \neq 0} \chi(x, k) = 1. \quad (1.6)$$

This function can be given as the solution of the integral equation [12]

$$\chi(x, k) = 1 + \int dx' G_0(x - x', k) u(x') \chi(x', k), \quad (1.7)$$

where  $G_0(x, k)$  is the Green’s function

$$G_0(x, k) = \frac{\operatorname{sgn} x_2}{2\pi i} \int d\alpha \theta(\alpha k_{\Im} x_2) e^{i\alpha x_1 - i\alpha(\alpha - 2k)x_2}, \quad k \in \mathbb{C}, \quad (1.8)$$

of the differential equation (1.1) with zero potential,

$$(i\partial_{x_2} + \partial_{x_1}^2 - 2ik\partial_{x_1})G_0(x, k) = \delta(x). \quad (1.9)$$

Solvability of these differential equations under some small norm assumptions was proved in [19] and thanks to (1.7) it is easy to show that  $\chi(x, k)$  has the asymptotic behavior

$$\lim_{|x_1| \rightarrow \infty} \chi(x, k) = 1 \quad (1.10)$$

on the  $x$ -plane,  $k_{\Im} \neq 0$ , independently on the direction. The Jost solution  $\Phi(x, k)$  defined by (1.5) obeys [17, 18] the following normalization and completeness conditions

$$\int dx_1 \overline{\Phi(x_1, x_2, k + p)} \Phi(x_1, x_2, \bar{k}) = 2\pi \delta(p), \quad p \in \mathbb{R}, \quad (1.11)$$

$$\int dk_{\Re} \overline{\Phi(x'_1, x_2, k)} \Phi(x_1, x_2, \bar{k}) = 2\pi \delta(x_1 - x'_1). \quad (1.12)$$

The Jost solution is an analytic function of  $k$  in the complex plane,  $k_{\Im} \neq 0$ . It has finite limits on the real axis,

$$\Phi^{\pm}(x, k) = \lim_{k_{\Im} \rightarrow \pm 0} \Phi(x, k). \quad (1.13)$$

By (1.11) these boundary values obey the normalization conditions

$$\int dx_1 \overline{\Phi^{\pm}(x_1, x_2, k)} \Phi^{\mp}(x_1, x_2, p) = 2\pi \delta(p - k), \quad p, k \in \mathbb{R}. \quad (1.14)$$

Spectral data are introduced as measure of the departure from analyticity of the Jost solutions. In literature there exists a variety of definitions of spectral data. In what follows we use the one suggested in [14, 17, 18]:

$$\mathcal{F}(k, p) = \frac{1}{2\pi} \int dx'_1 \overline{\Phi^+(x'_1, x_2, k)} \Phi^+(x'_1, x_2, p) - \delta(p - k), \quad p, k \in \mathbb{R}. \quad (1.15)$$

It is easy to check that  $\mathcal{F}(k, p)$  is independent on  $x_2$ . Condition of reality of the potential  $u$  in (1.1) is equivalent to the self-adjointness of the integral operator with kernel  $\mathcal{F}(k, p)$ . Now by (1.12) we have

$$\Phi^+(x, k) = \Phi^-(x, k) + \int dp \Phi^-(x, p) \mathcal{F}(p, k). \quad (1.16)$$

Thus the inverse scattering transform is the nonlocal Riemann–Hilbert problem of construction of the function  $\Phi(x, k)$  analytic in the upper and bottom half planes with normalization (1.5) and (1.6) and discontinuity at the real axis given by (1.16).

It was mentioned in [16, 3, 4, 5] that the integral equation (1.7) cannot be applied in the case of a potential  $u(x)$  not vanishing in all directions at large distances as the Green's function is slow decaying at space infinity. In [5] the following modification of this integral equation was suggested:

$$\chi(x, k) = 1 + \int_{-k_{\Im} \infty}^{x_1} dy_1 \int dx' \partial_{y_1} G_0(y_1 - x'_1, x_2 - x'_2, k) u(x') \chi(x', k), \quad (1.17)$$

where the order of operations is explicitly prescribed. Here and below we use notations of the type  $k_{\Im} \infty$  in the limits of integrals to indicate the sign of infinity. If the solution of this equation exists and is bounded, then like in the one dimensional case

$$\lim_{x_1 \rightarrow -k_{\Im} \infty} \chi(x, k) = 1, \quad k_{\Im} \neq 0, \quad (1.18)$$

while in contrast to (1.10) it can be different from 1 in the opposite direction. This modified integral equation is applicable to the simplest case of a potential of type (1.2), i.e. to the case  $u(x) = u(x_1)$  and it is trivial to check that it gives the standard (see [20]) one dimensional equation for the Jost solution. Nevertheless, the full description of the solutions of the Eq. (1.17) with potentials of the class (1.2) is absent till now. Only multiple pure soliton solutions were constructed in [21, 22]. In [23] and [5] it was shown that solutions of this equation can have additional cuts in the complex plane of the spectral parameter. Because of this we are studying here the special but rather wide subclass of potentials of type (1.2) that is obtained by applying recursively the so called binary Bäcklund transformations [24] with complex spectral parameter to a decaying potential. As we are interested in the spectral characteristics of potentials  $u$  having nontrivial limits, we study also the corresponding Darboux transformations furnishing the Jost solutions of the transformed potentials and their analytical properties as well as transformations of the spectral data.

We have at our disposal in [24] and [25] a rather simple and transparent method for performing binary Bäcklund transformations of the potential  $u$  and corresponding Darboux transformations of solutions of (1.1). Let  $\phi(x, k)$  be a solution of the nonstationary Schrödinger equation (1.4) with potential  $u(x)$ . Then the transformed potential is equal to

$$\tilde{u}(x) = u(x) - 2\partial_{x_1}^2 \log \Delta(x), \quad (1.19)$$

where

$$\Delta(x) = \int^{x_1} dx'_1 |\phi(x'_1, x_2, \lambda)|^2. \quad (1.20)$$

The Darboux transform of  $\phi(x, k)$ ,

$$\tilde{\phi}(x, k) = \phi(x, k) - \frac{\phi(x, \lambda)}{\Delta(x)} \int^{x_1} dx'_1 \overline{\phi(x'_1, x_2, \lambda)} \phi(x'_1, x_2, k), \quad (1.21)$$

solves the equation

$$(i\partial_{x_2} + \partial_{x_1}^2 - \tilde{u}(x))\tilde{\phi}(x, k) = 0 \quad (1.22)$$

with transformed potential. It is natural to expect that this transformation for complex parameter  $\lambda$  would supply an example of a potential of the type (1.2). Check of the fact that  $\tilde{\phi}$  obeys Eq. (1.22) is based on the following identity for a pair of arbitrary solutions  $f(x)$  and  $g(x)$  of the Eq. (1.22):

$$i\partial_{x_2}(\overline{f(x)}g(x)) = -\partial_{x_1}W(\overline{f(x)}, g(x)), \quad (1.23)$$

where Wronskian

$$W(\overline{f(x)}, g(x)) = \overline{f(x)}\partial_{x_1}g(x) - g(x)\partial_{x_1}\overline{f(x)} \quad (1.24)$$

was introduced.

In order to make equations (1.19)–(1.21) determined it is necessary to substitute indefinite integrals by definite ones and to choose constants of integration in a way that the potential  $\tilde{u}$  is real and regular. Thus in order to get transformations parametrized by constants and not by functions of  $x_2$  obeying some differential equations it is natural to choose integrals in (1.20)–(1.21) to be from infinity to  $x_1$ . Then we can use the fact that the asymptotic behavior (1.10) with respect to  $x_1$  is independent on  $x_2$ . On the other side, infinite limits of these integrals require exact control of their convergency in the recursive procedure. In addition one must write the solution  $\tilde{\phi}$  as a linear combination of the Jost solution and of a solution corresponding to discrete spectrum. The way to build the correct recursive procedures for generating both solutions is suggested by the remark that  $\phi(x, \lambda)\Delta(x)^{-1}$  is solution of the Eq. (1.22).

Thus we start with a regular rapidly decaying real potential  $u(x)$  for which all above mentioned elements of the direct and inverse problem are given. In Sec. 2 we introduce an exact recursion procedure for an arbitrary number of Bäcklund transformations and corresponding Darboux transformations for Jost solutions and solutions corresponding to the discrete spectrum. We formulate conditions of reality and regularity of the potentials constructed by these means and derive spectral data of the transformed Jost solutions (in analogy with [26], where transformations of the continuous spectra were considered). In Sec. 3 this recursion procedure is solved in terms of the original potential  $u(x)$  and its Jost solution  $\Phi(x, k)$ . By these means we get a solution depending on  $(N+1)N$  complex parameters describing  $N$  solitons superimposed to a generic background. Necessary and sufficient conditions satisfied by these parameters in order to get a regular and real solution are explicitly given. In the case  $u(x) \equiv 0$  we recover not only the multisoliton solutions obtained in [21], but we are able to identify all the regular and real solutions in the essentially more general class of multisoliton solutions derived in [22]. This extension based on the condition that the values of the Jost solution at the points of discrete spectrum are given as linear combinations of values of this solution in the conjugated points essentially complicates the whole construction, but the corresponding solutions are of the type essential for applications (see [9]–[10]). In Sec. 4 we present the

leading asymptotic behavior on the  $x$ -plane of the constructed potentials. We show that this behavior is indeed of the type (1.2) and that it essentially depends on the signs of the imaginary parts of parameters  $\lambda$  of the Bäcklund transformations. This essentially distinguishes the case of the two dimensional nonstationary Schrödinger equation from the case of the one dimensional stationary equation. In a forthcoming publication these results will be applied to the study of perturbations of such potentials, i.e. to the generic potentials of the type (1.2) by means of the formulation of the scattering problem on nontrivial background as suggested in [5].

## 2 Recursion procedure

Formulas (1.19), (1.20), and (1.21) enable us to formulate the recursion procedure for composing an arbitrary number of binary Bäcklund transformations. Indeed, if  $\phi_n(x, k)$  solves equation (1.4) with potential  $u_n(x)$ ,

$$i\partial_{x_2}\phi_n(x, k) + \partial_{x_1}^2\phi_n(x, k) = u_n(x)\phi_n(x, k), \quad (2.1)$$

then we specify Eq. (1.21) for  $\phi_{n+1}$  in the following way:

$$\phi_{n+1}(x, k) = \phi_n(x, k) - g_{n+1}(x) \left[ B'_{n+1}(k) + \int_{-(k_{\Im} + \lambda_{n+1}\Im)\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} \phi_n(x'_1, x_2, k) \right], \quad (2.2)$$

where we introduced notations

$$g_{n+1}(x) = \frac{\phi_n(x, \lambda_{n+1})}{\Delta_{n+1}(x)}, \quad (2.3)$$

$$\Delta_{n+1}(x) = c_{n+1} + \int_{-\lambda_{n+1}\Im\infty}^{x_1} dx'_1 |\phi_n(x'_1, x_2, \lambda_{n+1})|^2, \quad (2.4)$$

$$n = 0, 1, \dots,$$

and  $c_{n+1}$  and  $B'_{n+1}(k)$  are some  $x$ -independent constant and function of  $k$ , correspondingly. Then  $\phi_{n+1}$  has to be a solution of the shifted ( $n \rightarrow n+1$ ) Eq. (2.1) with potential

$$u_{n+1}(x) = u_n(x) - 2\partial_{x_1}^2 \log \Delta_{n+1}(x). \quad (2.5)$$

In what follows it is convenient to write all  $\phi_n$  as sums of two solutions of (2.1),

$$\phi_n(x, k) = F_n(x, k) + f_n(x, k), \quad (2.6)$$

that are given by the recursion relations

$$F_{n+1}(x, k) = F_n(x, k) - g_{n+1}(x) \int_{-(k_{\Im} + \lambda_{n+1}\Im)\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} F_n(x'_1, x_2, k), \quad (2.7)$$

$$f_{n+1}(x, k) = f_n(x, k) - g_{n+1}(x) \left[ B_{n+1}(k) + \int_{-\lambda_{n+1}\Im\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} f_n(x'_1, x_2, k) \right], \quad (2.8)$$

$$n = 0, 1, \dots,$$

where functions  $B_{n+1}(k)$  differ from  $B'_{n+1}(k)$  because of the different bottom limits of integrals in Eqs. (2.7) and (2.8). We also put

$$u_0(x) = u(x), \quad F_0(x, k) = \Phi(x, k), \quad f_0(x, k) = 0, \quad (2.9)$$

so that we start with the standard, real, regular, and rapidly decaying at space infinity potential  $u$ .

Properties of all these objects are given in the following

**Theorem 2.1** *Let the potential  $u(x)$  in (1.1) be real and rapidly decaying at space infinity and let  $\Phi(x, k)$  be the Jost solution of this equation, i.e. let  $\Phi(x, k)$  be defined by (1.5) and (1.7). Let also the sets of complex constants  $\lambda_1, \lambda_2, \dots$ , real nonzero constants  $c_1, c_2, \dots$  obey the following conditions:*

$$\lambda_{n\Im} \neq 0, \quad n = 1, 2, \dots, \quad (2.10)$$

$$|\lambda_{1\Im}| > |\lambda_{2\Im}| > \dots, \quad (2.11)$$

$$\lambda_{n\Im} c_n > 0, \quad n = 1, 2, \dots, \quad (2.12)$$

and let be given some functions  $B_1(k), B_2(k), \dots$  of the complex parameter  $k$ .

1. Integrals in (2.7), (2.8), and (2.4) converge and define regular functions of  $x$  and  $k$ , for  $k_{\Im} \neq 0, k_{\Im} + \lambda_{j\Im} \neq 0$  ( $j = 1, 2, \dots, n$ ). Functions  $\Delta_{n+1}(x)$  have no zeroes on the  $x$ -plane.
2. There exist nonzero limits

$$\lim_{x_1 \rightarrow \pm k_{\Im} \infty} e^{ikx_1 + ik^2 x_2} F_n(x, k) = A_n(\pm, k) \quad (2.13)$$

that are independent of  $x_2$  and obey the recursion relation

$$A_0(\pm, k) = 1, \quad (2.14)$$

$$A_{n+1}(\pm, k) = \left[ 1 + \theta(\pm k_{\Im} \lambda_{n+1\Im}) \frac{2i\lambda_{n+1\Im}}{\lambda_{n+1} - k} \right] A_n(\pm, k). \quad (2.15)$$

3. There exist finite nonzero limits

$$\begin{aligned} & \lim_{|x_1| \rightarrow \infty} e^{i\lambda_{n+1\Re} x_1 + |\lambda_{n+1\Im} x_1|} g_{n+1}(x) = \\ & = \begin{cases} \frac{2\lambda_{n+1\Im}}{A_n(+, \lambda_{n+1})} e^{-i\bar{\lambda}_{n+1}^2 x_2}, & x_1 \rightarrow +\lambda_{n+1\Im} \infty, \\ \frac{A_n(-, \lambda_{n+1})}{c_{n+1}} e^{-i\lambda_{n+1}^2 x_2}, & x_1 \rightarrow -\lambda_{n+1\Im} \infty, \end{cases} \end{aligned} \quad (2.16)$$

4. There exist finite nonzero limits (for  $n \geq 1$ )

$$\begin{aligned} & \lim_{|x_1| \rightarrow \infty} e^{i\lambda_{n\Re} x_1 + |\lambda_{n\Im} x_1|} f_n(x, k) = \\ & = - \begin{cases} (B_n(k) + b_n(k)) \frac{2\lambda_{n\Im}}{A_{n-1}(+, \lambda_n)} e^{-i\bar{\lambda}_n^2 x_2}, & x_1 \rightarrow +\lambda_{n\Im} \infty, \\ B_n(k) \frac{A_{n-1}(-, \lambda_n)}{c_n} e^{-i\lambda_n^2 x_2}, & x_1 \rightarrow -\lambda_{n\Im} \infty, \end{cases} \end{aligned} \quad (2.17)$$

where the functions

$$b_n(k) = \operatorname{sgn} \lambda_{n\Im} \int_{-\infty}^{+\infty} dx'_1 \overline{\phi_{n-1}(x'_1, x_2, \lambda_n)} f_{n-1}(x'_1, x_2, k) \quad (2.18)$$

are  $x_2$ -independent.

5. Functions  $F_n(x, k)$ ,  $g_n(x)$ ,  $f_n(x, k)$ , and  $\phi_n(x, k)$  solve the nonstationary Schrödinger equation (2.1) with potential

$$u_n(x) = u(x) - 2\partial_{x_1}^2 \log \prod_{j=1}^n \Delta_j(x). \quad (2.19)$$

6. Functions

$$\Phi_n(x, k) = \frac{F_n(x, k)}{A_n(-, k)}, \quad n = 1, 2, \dots, \quad (2.20)$$

are the Jost solutions of the Eq. (2.1) with potential (2.5), i.e.

$$\chi_n(x, k) = e^{ikx_1 + ik^2 x_2} \Phi_n(x, k) \quad (2.21)$$

obey the modified integral equations (1.17) with  $u_n(x)$  from (2.19) substituted for  $u(x)$ .

The proof of the theorem is by induction on  $n$ . Therefore, it will be sometime useful in referring to a formula  $(\#)$  depending on  $n$  to make explicit this dependence by writing  $(\#)_n$  and then  $(\#)_{n+1}$  when the same formula is considered for  $n \rightarrow n+1$ .

We divide the proof into a sequence of Lemmas.

**Lemma 2.1** *Let for some  $n \geq 1$  the functions  $F_n$  and  $f_n$  be regular functions of  $x$  and  $k$ , for  $k_{\Im} \neq 0$ ,  $k_{\Im} + \lambda_{j\Im} \neq 0$  ( $j = 1, 2, \dots, n$ ) and obey statements 2 and 4 of the theorem, or let they be given by (2.9) for  $n = 0$ . Let the functions  $B_n(k)$  be regular functions of  $k$  and  $\lambda_n$ ,  $\lambda_{n+1}$ ,  $c_{n+1}$  obey (2.10)–(2.12). Then for  $\phi_n$  defined in (2.6) there exists the limit*

$$\lim_{x_1 \rightarrow \pm k_{\Im} \infty} e^{ikx_1 + ik^2 x_2} \phi_n(x, k) = A_n(\pm, k), \quad k \in \mathbb{C}, \quad |k_{\Im}| < |\lambda_{n\Im}|, \quad (2.22)$$

where  $A_n$  is given in (2.13) and  $\Delta_{n+1}$  determined by (2.4) exists, has no zeroes on the  $x$ -plane and obeys the asymptotic behavior

$$\Delta_{n+1}(x) \rightarrow \begin{cases} \frac{|A_n(+, \lambda_{n+1})|^2}{2\lambda_{n+1\Im}} e^{2\lambda_{n+1\Im}(x_1 + 2\lambda_{n+1\Re}x_2)}, & x_1 \rightarrow +\lambda_{n+1\Im}\infty, \\ c_{n+1}, & x_1 \rightarrow -\lambda_{n+1\Im}\infty, \end{cases} \quad (2.23)$$

where in the case  $n = 0$  in agreement with (2.6)  $A_0 = 1$ .

*Proof.* In the case  $n = 0$  (2.22) is nothing but the direct consequence of (1.7) and (2.9). If  $n \geq 1$  then this asymptotic behavior of  $\phi_n$  trivially follows from (2.6), (2.13) and (2.17). Thanks to this asymptotic behavior taking into account that  $|\lambda_{n\Im}| > |\lambda_{n+1\Im}| > 0$  we get convergency of the integral in (2.4). Asymptotics (2.23) follows from (2.22). Taking into account that  $\Delta_{n+1}$  by definition is a monotonous function of  $x_1$  and that thanks to (2.12) the signs of both asymptotic limits in (2.23) coincide, we see that this function has no zeroes on the  $x$ -plane.

**Lemma 2.2** *Under conditions of Lemma 2.1 function  $g_{n+1}$  as defined by (2.3) is regular and obeys the asymptotic properties (2.16).*

*Proof.* Proof of this Lemma follows directly from results of Lemma 2.1 as  $\Delta_{n+1}$  has no zeroes and the asymptotic behaviors (2.22), (2.23) guarantee that  $g_{n+1}$  decays with proper exponent for growing  $x$ . The exact values of the limits are also readily obtained from (2.22) and (2.23).

**Lemma 2.3** *Under conditions of Lemma 2.1 the function  $f_{n+1}$  as defined by (2.8) is regular and obeys the asymptotic properties (2.17) $_{n+1}$  and (2.18) $_{n+1}$ .*

*Proof.* Convergency of the integral in (2.8) and regularity of  $f_{n+1}$  follow directly from Lemma 2.1 if we take into account that by (2.11)  $|\lambda_{n+1\Im}| < |\lambda_{n\Im}|$ . In fact, thanks to this inequality, by (2.22) $_n$  and (2.17) $_n$  the integrals  $\int_{\pm\infty}^{x_1} \phi_n(x'_1, x_2, \lambda_{n+1}) f_n(x'_1, x_2, k)$  are convergent. Then from (2.8) we get

$$e^{i\lambda_{n+1\Re}x_1 + |\lambda_{n+1\Im}x_1|} f_{n+1}(x, k) = e^{i\lambda_{n+1\Re}x_1 + |\lambda_{n+1\Im}x_1|} f_n(x, k) - e^{i\lambda_{n+1\Re}x_1 + |\lambda_{n+1\Im}x_1|} g_{n+1}(x) \left[ B_{n+1}(k) + \int_{-\lambda_{n+1\Im}\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} f_n(x'_1, x_2, k) \right].$$

The asymptotic property (2.17) $_n$  guarantees that the first term in the r.h.s. goes to zero when  $|x_1| \rightarrow \infty$  because of condition  $|\lambda_{n+1\Im}| < |\lambda_{n\Im}|$ . The first factor of the second term has finite limits by Lemma 2.2 and thanks to the mentioned convergency of the integral in brackets we prove (2.17) $_{n+1}$  and (2.18) $_{n+1}$ .

**Lemma 2.4** *Under conditions of Lemma 2.1  $F_{n+1}$  as defined by (2.7) is a regular function obeying asymptotic property (2.13) $_{n+1}$  and  $A_{n+1}$  is given by means of (2.15).*

*Proof.* Convergency of the integral in (2.7) follows from the asymptotic behaviors (2.13) $_n$  and (2.22) $_n$  and then regularity of  $F_{n+1}$  from Lemma 2.2. In order to prove (2.13) $_{n+1}$ , first, we consider the asymptotic behavior of the integral term:

$$e^{i(k - \bar{\lambda}_{n+1})x_1 + i(k^2 - \bar{\lambda}_{n+1}^2)x_2} \int_{-(k_{\Im} + \lambda_{n+1\Im})\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} F_n(x'_1, x_2, k) \rightarrow \overline{A_n(\pm \operatorname{sgn}(k_{\Im} \lambda_{n+1\Im}), \lambda_{n+1})} \frac{A_n(\pm, k)}{i(\bar{\lambda}_{n+1} - k)}, \quad x_1 \rightarrow \pm k_{\Im} \infty, \quad (2.24)$$

where (2.13) $_n$  and (2.22) $_n$  were used. Let us now write (2.7) as

$$e^{ikx_1 + ik^2x_2} F_{n+1}(x, k) = e^{ikx_1 + ik^2x_2} F_n(x, k) - e^{\lambda_{n+1\Re}x_1 - |\lambda_{n+1\Im}x_1|} \left( e^{i\lambda_{n+1\Re}x_1 + |\lambda_{n+1\Im}x_1| + i\bar{\lambda}_{n+1}^2x_2} g_{n+1}(x) \right) \times \left( e^{i(k - \bar{\lambda}_{n+1})x_1 + i(k^2 - \bar{\lambda}_{n+1}^2)x_2} \int_{-(k_{\Im} + \lambda_{n+1\Im})\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} F_n(x'_1, x_2, k) \right).$$

In the first term of the r.h.s. we can use (2.13) $_n$ . The second term is given as a product of three multipliers, each of them having finite limit for  $x_1$  going to infinity. Moreover, the first of them has nonzero limit only for  $x_1 \rightarrow \lambda_{n+1\Im}\infty$ . Then (2.15) $_{n+1}$  follows from (2.16) $_n$  proved in Lemma 2.2 and the asymptotic limit of the last multiplier given above.



**Lemma 2.5** *If under the assumptions of Lemma 2.1 statement 5 of Theorem 2.1 is satisfied then this statement is also valid for  $n \rightarrow n+1$  with potential  $u_{n+1}$  given in  $(2.19)_{n+1}$ .*

*Proof.* By the inductive hypothesis  $\phi_n$  obeys

$$(i\partial_{x_2} + \partial_{x_1}^2 - u_n(x))\phi_n(x, k) = 0 \quad (2.25)$$

and the same equation is valid for  $F_n$ ,  $g_n$ , and  $f_n$ . Thus by identity (1.23)

$$i\partial_{x_2} \int_{-(k_{\Im} + \lambda_{n+1\Im})\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} F_n(x'_1, x_2, k) = -W(\overline{\phi_n(x, \lambda_{n+1})}, F_n(x, k)),$$

where the asymptotic behaviors (2.13) and (2.22) were taken into account. Analogously from (2.4) we get

$$i\partial_{x_2} \Delta_{n+1}(x) = -W(\overline{\phi_n(x, \lambda_{n+1})}, \phi_n(x, \lambda_{n+1})). \quad (2.26)$$

Then by (2.3) and (2.25) we get

$$(i\partial_{x_2} + \partial_{x_1}^2 - u_{n+1}(x))g_{n+1}(x) = 0 \quad (2.27)$$

with potential  $u_{n+1}(x)$  defined as in (2.5). Now, by (2.6), (2.7) and (2.8) it is easy to check that  $F_{n+1}$ ,  $f_{n+1}$ , and  $\phi_{n+1}$  solve (2.27).

**Lemma 2.6** *If under the assumptions of Lemma 2.1 statement 5 of Theorem 2.1 is fulfilled then statement 6 is valid for  $n \rightarrow n+1$ .*

*Proof.* We have to demonstrate that  $\chi_{n+1}$  as defined by (2.20) and (2.21) obeys integral equation (1.17) with  $u_{n+1}$  given in (2.5). Since we know from Lemma 2.5 that  $F_{n+1}$  obeys differential equation (2.27), we can write

$$\begin{aligned} \int dx' \partial_{x_1} G_0(x - x', k) u_{n+1}(x') \chi_{n+1}(x', k) &= \\ &= \int dx' \partial_{x_1} G_0(x - x', k) (i\partial_{x'_2} + \partial_{x'_1}^2 - 2ik\partial_{x'_1}) \chi_{n+1}(x', k). \end{aligned}$$

Taking into account that the  $\partial_{x_1}$  derivative cancels the slowly decaying terms in the asymptotic behavior of the Green's function we can integrate by parts and then use (1.9). Thus

$$\begin{aligned} 1 + \int_{-k_{\Im}\infty}^{x_1} dy_1 \int dx' \partial_{y_1} G_0(y_1 - x'_1, x_2 - x'_2, k) u_{n+1}(x') \chi_{n+1}(x', k) &= \\ &= 1 + \int_{-k_{\Im}\infty}^{x_1} dy_1 \partial_{y_1} \chi_{n+1}(y_1, x_2, k) = \\ &= 1 + \chi_{n+1}(x, k) - \lim_{x_1 \rightarrow -k_{\Im}\infty} \chi_{n+1}(x, k). \end{aligned}$$

This proves the lemma thanks to (2.13), (2.20) and (2.21). In the same way it is easy to show that  $g_n(x)$  and  $f_n(x, k)$  obey the corresponding homogeneous integral equation.

*Proof of the Theorem 2.1* now follows from Lemmas 2.1–2.6 by induction on  $n$  since thanks to (2.9) the theorem is valid for  $n = 0$ .

**Corollary 2.1** *We have scalar products*

$$\int dx'_1 \overline{F_n(x'_1, x_2, k+p)} F_n(x'_1, x_2, \bar{k}) = 2\pi\delta(p), \quad p \in \mathbb{R}, \quad (2.28)$$

$$\int dx'_1 \overline{F_n(x'_1, x_2, k+p)} f_n(x'_1, x_2, \bar{k}) = 0. \quad (2.29)$$

*Proof.* By means of (2.7) we have

$$\begin{aligned} \overline{F_{n+1}(x, k+p)} F_{n+1}(x, \bar{k}) &= \overline{F_n(x, k+p)} F_n(x, \bar{k}) - \\ &- \partial_{x_1} \left[ \frac{1}{\Delta_{n+1}(x)} \int_{-(k_{\Im} + \lambda_{n+1}\Im)\infty}^{x_1} dx'_1 \phi_n(x'_1, x_2, \lambda_{n+1}) \overline{F_n(x'_1, x_2, k+p)} \times \right. \\ &\times \left. \int_{(k_{\Im} - \lambda_{n+1}\Im)\infty}^{x_1} dx'_1 \overline{\phi_n(x'_1, x_2, \lambda_{n+1})} F_n(x'_1, x_2, \bar{k}) \right] \end{aligned}$$

It is easy to see that the integral of the last term is equal to zero and, therefore, we get

$$\int dx'_1 \overline{F_{n+1}(x'_1, x_2, k+p)} F_{n+1}(x'_1, x_2, \bar{k}) = \int dx'_1 \overline{F_n(x'_1, x_2, k+p)} F_n(x'_1, x_2, \bar{k}) \quad (2.30)$$

that proves (2.28) thanks to (1.11) and (2.9). The second equality is derived analogously.

**Corollary 2.2**  *$F_n(x, k)$  is an analytic function in the complex plane of  $k$  with possible discontinuity at the real axis and poles at points  $k = \bar{\lambda}_j$ ,  $j = 1, \dots, n$ .*

*Proof.* Indeed, by definition (2.7) we see that  $F_{n+1}$  inherits analyticity properties of  $F_n$  and has an additional discontinuity at  $k_{\Im} = -\lambda_{n+1}\Im$ . Thanks to (2.7) and (2.28), (2.29) we have that

$$\begin{aligned} F_{n+1}(x, k_{\Re} - i(\lambda_{n+1}\Im + 0)) - F_{n+1}(x, k_{\Re} - i(\lambda_{n+1}\Im - 0)) &= \\ &= 2\pi\delta(k_{\Re} - \lambda_{n+1}\Re)g_{n+1}(x). \end{aligned} \quad (2.31)$$

Thus we see, that  $F_{n+1}(x, k)$  has an additional pole at  $k = \bar{\lambda}_{n+1}$ ,

$$F_{n+1}(x, k) = -\frac{ig_{n+1}(x)}{k - \bar{\lambda}_{n+1}} + O(1), \quad k \rightarrow \bar{\lambda}_{n+1}. \quad (2.32)$$

**Corollary 2.3** *Because of (2.14) and (2.15)*

$$A_n(\pm, k) = \prod_{j=1}^n \left( \frac{k - \lambda_j}{k - \bar{\lambda}_j} \right)^{\theta(\pm k_{\Im} \lambda_{j\Im})}. \quad (2.33)$$

Thus  $A_n(-, k)$  is a meromorphic function discontinuous on the real axis of  $k$  that has simple poles at  $k = \bar{\lambda}_j$ ,  $j = 1, \dots, n$ .

**Corollary 2.4** *Let us introduce the transmission coefficients*

$$a_n(k) = \frac{A_n(+, k)}{A_n(-, k)} = \prod_{j=1}^n \left( \frac{k - \lambda_j}{k - \bar{\lambda}_j} \right)^{\text{sgn } k_{\Im} \lambda_{j\Im}}. \quad (2.34)$$

*Then like in the one-dimensional case*

$$a_n(k) = \lim_{x_1 \rightarrow +k_{\Im} \infty} \chi_n(x, k). \quad (2.35)$$

*This function is analytic in the upper and bottom half planes with a discontinuity at the real axis and has simple zeroes at points  $k = \lambda_j$  and  $k = \bar{\lambda}_j$ ,  $j = 1, \dots, n$ .*

**Corollary 2.5**  $\Phi_n(x, k)$  *is an analytic function in the complex plane of  $k$  with possible discontinuity at the real axis and*

$$\Phi_{n+1}(x, \bar{\lambda}_{n+1}) = -\frac{g_{n+1}(x)}{2\lambda_{n+1\Im}} \prod_{j=1}^n \left( \frac{\bar{\lambda}_{n+1} - \bar{\lambda}_j}{\bar{\lambda}_{n+1} - \lambda_j} \right)^{\theta(\lambda_{n+1\Im} \lambda_{j\Im})}. \quad (2.36)$$

*Proof follows from Corollaries 2.2 and 2.3 since the singularities of  $F_n(x, k)$  in the complex plane are compensated by normalization (2.20). Then (2.36) follows from (2.32).*

**Corollary 2.6** *Let the boundary values of  $\Phi_n(x, k)$  and  $a_n(k)$  at the real axis be defined in analogy with (1.13). Then for these values we have relation*

$$\frac{\Phi_n^+(x, k)}{a_n^+(k)} = \Phi_n^-(x, k) + \int dp \Phi_n^-(x, p) \mathcal{F}_n(p, k). \quad (2.37)$$

*(cf. (1.16)), where continuous part of the spectral data is given by*

$$\mathcal{F}_n(k, p) = \mathcal{F}(k, p) \prod_{j=1}^n \left( \frac{(k - \lambda_j)(p - \bar{\lambda}_j)}{(k - \bar{\lambda}_j)(p - \lambda_j)} \right)^{\theta(\lambda_{j\Im})}, \quad k, p \in \mathbb{R}. \quad (2.38)$$

*Proof.* For the boundary values of  $F_n(x, k)$  at the real axis in analogy with (2.30) we derive

$$\begin{aligned} \int dx'_1 \overline{F_{n+1}^{\mp}(x'_1, x_2, k)} F_{n+1}^{\pm}(x'_1, x_2, p) &= \int dx'_1 \overline{F_n^{\mp}(x'_1, x_2, k)} F_n^{\pm}(x'_1, x_2, p) = \\ &= 2\pi\delta(k - p) \end{aligned} \quad (2.39)$$

$$\begin{aligned} \int dx'_1 \overline{F_{n+1}^+(x'_1, x_2, k)} F_{n+1}^+(x'_1, x_2, p) &= \int dx'_1 \overline{F_n^+(x'_1, x_2, k)} F_n^+(x'_1, x_2, p) = \\ &= 2\pi\mathcal{F}(k, p) + 2\pi\delta(k - p), \end{aligned} \quad (2.40)$$

where (1.14) and (1.15) were used. Then (2.37) follows from definition (2.20) and (2.33).

### 3 Resolution of the recursion relations

In order to resolve the recursion relations explicitly we introduce

$$B_{l,m}(x) = \int_{-(\lambda_{l\Im} + \lambda_{m\Im})\infty}^{x_1} dx'_1 \overline{\Phi(x'_1, x_2, \lambda_l)} \Phi(x'_1, x_2, \lambda_m), \quad (3.1)$$

$$\beta_l(x, k) = \int_{-(k_{\Im} + \lambda_{l\Im})\infty}^{x_1} dx'_1 \overline{\Phi(x'_1, x_2, \lambda_l)} \Phi(x'_1, x_2, k), \quad (3.2)$$

$$l, m = 1, 2, \dots,$$

so that

$$B_{l,m}(x) = \beta_l(x, \lambda_m) = \overline{\beta_m(x, \lambda_l)}. \quad (3.3)$$

Let  $B_n(x)$  denotes the  $n \times n$  matrix

$$B_n(x) = \|B_{l,m}(x)\|_{l,m=1,\dots,n} \quad (3.4)$$

and let us define the following row and two columns

$$\Phi(x) = (\Phi(x, \lambda_1), \dots, \Phi(x, \lambda_n)), \quad (3.5)$$

$$\beta(x, k) = (\beta_1(x, k), \dots, \beta_n(x, k))^T, \quad (3.6)$$

$$\gamma(x, k) = (\gamma_1(k), \dots, \gamma_n(k))^T, \quad (3.7)$$

where subscript T means transposition and  $\gamma_n(k)$  are some given functions such that matrix

$$C_n = \|c_{l,m}\|_{l,m=1,\dots,n}, \quad c_{l,m} = \gamma_l(\lambda_m), \quad (3.8)$$

is Hermitian. Let us denote

$$A_n(x) = C_n + B_n(x), \quad (3.9)$$

that is also Hermitian matrix by construction.

In order to formulate conditions of regularity of the potential  $u_n(x)$  we introduce also matrices

$$C_n(y) = \|\{c_{l,m} | y\lambda_{l\Im} > 0, y\lambda_{m\Im} > 0\}\|_{l,m=1,\dots,n}, \quad (3.10)$$

where  $y$  is some real parameter, in fact its sign. Their sizes can be less than  $n \times n$ , as they are constructed by removing from matrix  $C_n$  those rows and columns that do not obey the inequalities in (3.10). These matrices are Hermitian also and if all rows and columns are removed, then we put by definition  $\det C_n(y) = 1$ . Let also  $\Lambda_n(y)$  denote matrix with entries  $-i(\bar{\lambda}_l - \lambda_m)^{-1}$  obeying the same properties as in (3.10). One can show that

$$\det \Lambda_n(y) = \prod_{l=1}^n (2\lambda_{l\Im})^{-\theta(y\lambda_{l\Im})} \prod_{\substack{l,m=1 \\ l \neq m}}^n \left| \frac{\lambda_l - \lambda_m}{\bar{\lambda}_l - \lambda_m} \right|^{\theta(y\lambda_{l\Im})\theta(y\lambda_{m\Im})}. \quad (3.11)$$

In these terms we impose the following conditions on the constants  $c_{l,m}$ :

$$\pm C_n(\pm) > 0 \quad (3.12)$$

and prove the following

**Theorem 3.1** *Let conditions (2.10), (2.11), and (3.12) be fulfilled. Then for any  $n = 1, 2, \dots$  and  $x$*

$$\det A_n(x) \prod_{l=1}^n \lambda_{l\Im} > 0, \quad \det A_0(x) = 1, \quad (3.13)$$

and the solutions of the recursion equations (2.5)–(2.9) are given by means of the following relations:

$$\Delta_n(x) = \frac{\det A_n(x)}{\det A_{n-1}(x)}, \quad (3.14)$$

$$F_n(x, k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \beta(x, k) \\ \Phi(x) & \Phi(x, k) \end{vmatrix}, \quad (3.15)$$

$$f_n(x, k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \gamma(k) \\ \Phi(x) & 0 \end{vmatrix}, \quad (3.16)$$

$$g_n(x) = \frac{-1}{\det A_n(x)} \begin{vmatrix} A_n(x) & e_n \\ \Phi(x) & 0 \end{vmatrix}, \quad e_n = (0, \dots, 0, 1)^T, \quad (3.17)$$

$$u_n(x) = u(x) - 2\partial_{x_1}^2 \log \det A_n(x), \quad (3.18)$$

$$n = 1, 2, \dots$$

Moreover, for the constants  $c_n$  in (2.4) we have relation

$$c_{n+1} = \frac{\det C_{n+1}(\lambda_{n+1\Im})}{\det C_n(\lambda_{n+1\Im})}, \quad (3.19)$$

where the matrices  $C_n(\pm)$  are defined in (3.12).

In order to prove that these relations resolve (2.5)–(2.9) we need to calculate all involved integrals. By (2.6), (3.15) and (3.16)

$$\phi_n(x, k) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & \gamma(k) + \beta(x, k) \\ \Phi(x) & \Phi(x, k) \end{vmatrix}, \quad (3.20)$$

so that

$$\phi_n(x, \lambda_{n+1}) = \frac{1}{\det A_n(x)} \begin{vmatrix} A_n(x) & A_{*,n+1}(x) \\ \Phi(x) & \Phi(x, \lambda_{n+1}) \end{vmatrix}, \quad (3.21)$$

where we introduced the column

$$A_{*,n+1}(x) = (A_{1,n+1}(x), \dots, A_{n,n+1}(x))^T \quad (3.22)$$

and used (3.3), (3.8), and (3.9) in order to write  $\gamma_m(\lambda_{n+1}) + \beta_m(x, \lambda_{n+1}) = A_{m,n+1}(x)$ . Let us also introduce the matrix

$$A_{n+1}(x, k) = \begin{pmatrix} A_n(x) & \gamma(k) + \beta(x, k) \\ A_{n+1,*}(x) & \beta_{n+1}(x, k) \end{pmatrix}, \quad (3.23)$$

where the row  $A_{n+1,*}(x)$  is the transposition of the column (3.22). Below for any matrix  $A$  we denote as  $A^{(k,l)}$  the same matrix with removed  $l$ -th row and  $m$ -th column, say,

$$A_{n+1}^{(l,m)}(x) = \|(A_{n+1}(x, k))_{i,j} \|_{\substack{i,j=1,\dots,n, \\ i \neq l, \quad j \neq m}}. \quad (3.24)$$

**Lemma 3.1** *Let Theorem 3.1 be valid for some  $n$ . Then*

$$\overline{\phi_n(x, \lambda_{n+1})} \phi_n(x, k) = \partial_{x_1} \frac{\det A_{n+1}(x, k)}{\det A_n(x)}. \quad (3.25)$$

*Proof.* Thanks to notation (3.23) we can write the expansions of (3.21) and (3.20) with respect to the last row as

$$\overline{\phi_n(x, \lambda_{n+1})} = \frac{1}{\det A_n(x)} \sum_{k=1}^{n+1} (-1)^{n+1+k} \overline{\Phi(x, \lambda_k)} \det A_{n+1}^{(k, n+1)}(x, k), \quad (3.26)$$

$$\begin{aligned} \phi_n(x, k) &= \frac{1}{\det A_n(x)} \left[ \sum_{l=1}^n (-1)^{n+l+1} \Phi(x, \lambda_l) \det A_{n+1}^{(n+1, l)}(x, k) + \right. \\ &\quad \left. + \Phi(x, k) \det A_{n+1}^{(n+1, n+1)}(x) \right]. \end{aligned} \quad (3.27)$$

Then taking into account (3.1), (3.3), (3.9), and (3.23) we get that  $\overline{\Phi(x, \lambda_m)} \Phi(x, \lambda_l) = \partial_{x_1}(A_{n+1}(x, k))_{m, l}$ ,  $l \leq n$ , and  $\overline{\Phi(x, \lambda_m)} \Phi(x, k) = \partial_{x_1}(A_{n+1}(x, k))_{m, n+1}$ . Thus

$$\begin{aligned} \overline{\phi_n(x, \lambda_{n+1})} \phi_n(x, k) &= \\ &= \sum_{k, l=1}^{n+1} (-1)^{k+l} \frac{(\partial_{x_1} A_{n+1}(x, k))_{k, l}}{\det^2 A_n(x)} \det A_{n+1}^{(k, n+1)}(x, k) \det A_{n+1}^{(n+1, l)}(x, k) \end{aligned}$$

and the statement of the lemma follows from the known property of determinants: If  $A_n$ ,  $n = 1, 2, \dots$ , are  $n \times n$  matrices depending on some parameter and such that for every  $n$  the matrix  $A_n$  is just the upper main minor of matrix  $A_{n+1}$  then we have the identity

$$\begin{aligned} (\det A_{n+1})' \det A_n - (\det A_n)' \det A_{n+1} &= \\ &= \sum_{k, l=1}^{n+1} (-1)^{k+l} A'_{k, l} \det A_{n+1}^{(k, n+1)} \det A_{n+1}^{(n+1, l)}, \end{aligned} \quad (3.28)$$

where prime denotes derivative with respect to this parameter.

**Lemma 3.2** *Let  $A_n$  be defined in (3.9). Then the leading asymptotic behavior of its determinant for  $|x_1| \rightarrow \infty$  and  $x_2$  fixed is given by*

$$\det A_n(x) = \det C_n(\mp) \det \Lambda_n(\pm) \prod_{j=1}^n \left| 1 + e^{-i\lambda_j x_1 - i\lambda_j^2 x_2} \right|^2, \quad x_1 \rightarrow \pm\infty, \quad (3.29)$$

where matrices  $C_n(\mp)$  and  $\det \Lambda_n(\pm)$  are defined in (3.10) and (3.11).

*Proof.* Let us introduce the  $n \times n$  diagonal matrix

$$D_n(x) = \text{diag} \left\{ 1 + e^{-i\lambda_j x_1 - i\lambda_j^2 x_2} \right\}_{j=1}^n. \quad (3.30)$$

Then, taking into account that thanks to (1.5), (1.10), and (3.2)

$$\begin{aligned} e^{i(k - \bar{\lambda}_l)x_1 + i(k^2 - \bar{\lambda}_l^2)x_2} \beta_l(x, k) &\rightarrow \frac{1}{i(\bar{\lambda}_l - k)} \\ e^{i(\lambda_m - \bar{\lambda}_l)x_1 + i(\lambda_m^2 - \bar{\lambda}_l^2)x_2} B_{l, m}(x) &\rightarrow \frac{1}{i(\bar{\lambda}_l - \lambda_m)} \end{aligned} \quad (3.31)$$

for  $|x_1| \rightarrow \infty$  we derive that

$$\lim_{x_1 \rightarrow \pm\infty} \overline{D_n(x)}^{-1} A_n(x) D_n(x)^{-1} = \alpha_n(\pm), \quad (3.32)$$

where the  $n \times n$  matrices  $\alpha_n(\pm) = \|\alpha_{l,m}(\pm)\|_{l,m=1,\dots,n}$ ,  $\alpha_0 = 1$ , are defined by means of their entries

$$\alpha_{l,m}(\pm) = \theta(\lambda_{l\Im}\lambda_{m\Im}) \left( c_{l,m}\theta(\mp\lambda_{l\Im}) + \frac{\theta(\pm\lambda_{l\Im})}{i(\bar{\lambda}_l - \lambda_m)} \right). \quad (3.33)$$

We see that due to (3.33) only entries  $(l, m)$  in the l.h.s. obeying condition  $\lambda_{l\Im}\lambda_{m\Im} > 0$  can give nontrivial limits. It is easy to notice that entries such that  $\lambda_{l\Im}\lambda_{m\Im} < 0$  are decaying at least as  $e^{-|\lambda_{n\Im}x_1|}$ , i.e. by (2.11) as the lowest exponential involved in the matrix  $A_n$ .

Taking into account the block structure of the matrix  $\alpha_n(\pm)$  we get by (3.12)

$$\det \alpha_n(\pm) = \det C_n(\mp) \det \Lambda_n(\pm),$$

that gives (3.29).

**Lemma 3.3** *Let Theorem 3.1 be valid for some  $n$  and let  $\lambda_1, \dots, \lambda_{n+1}$  obey (2.11). Then for  $\Delta_{n+1}(x)$  defined in (2.4) we have equality (3.14) <sub>$n+1$</sub>  where the coefficient  $c_{n+1,n+1}$  of the matrix  $A_{n+1}(x)$  is given by*

$$c_{n+1,n+1} = c_{n+1} + C_{n+1,*}(\lambda_{n+1\Im})C_n^{-1}(\lambda_{n+1\Im})C_{*,n+1}(\lambda_{n+1\Im}), \quad (3.34)$$

where  $C_{n+1,*}(\lambda_{n+1\Im})$  and  $C_{*,n+1}(\lambda_{n+1\Im})$  are the last row and column of  $C_{n+1}(\lambda_{n+1\Im})$ . Moreover, matrix  $A_{n+1}(x)$  obeys property (3.13) <sub>$n+1$</sub> .

*Proof.* In order to calculate the integral in (2.4) we use (3.21), so  $\gamma_m(\lambda_{n+1}) = c_{m,n+1}$ . Then by Lemma 3.1

$$\overline{\phi_n(x, \lambda_{n+1})} \phi_n(x, \lambda_{n+1}) = \partial_{x_1} \frac{\det A_{n+1}(x)}{\det A_n(x)},$$

where we used that in this case  $\det A_{n+1}(x, \lambda_{n+1}) = \det A_{n+1}(x) - c_{n+1,n+1} \det A_n(x)$  by (3.23). Then by Lemma 3.2 we see that the integral in (2.4) is convergent and that

$$\Delta_{n+1}(x) = c_{n+1} + \frac{\det A_{n+1}(x)}{\det A_n(x)} - \frac{\det C_{n+1}(\lambda_{n+1\Im}) \det \Lambda_{n+1}(-\lambda_{n+1\Im})}{\det C_n(\lambda_{n+1\Im}) \det \Lambda_n(-\lambda_{n+1\Im})}.$$

By definition (3.11)

$$\Lambda_{n+1}(-\lambda_{n+1\Im}) = \Lambda_n(-\lambda_{n+1\Im}), \quad (3.35)$$

thus the determinants of  $\Lambda$  cancel out and (3.14) <sub>$n+1$</sub>  follows by (3.19). The latter one is equivalent to (3.34) thanks to the following property of the determinants of bordered matrices:

$$\frac{1}{\det C_n} \begin{vmatrix} C_n & C_{*,n+1} \\ C_{n+1,*} & c_{n+1,n+1} \end{vmatrix} = c_{n+1,n+1} - C_{n+1,*}C_n^{-1}C_{*,n+1}. \quad (3.36)$$

In order to prove (3.13) <sub>$n+1$</sub>  let us mention first that by (3.10)

$$C_{n+1}(-\lambda_{n+1\Im}) = C_n(-\lambda_{n+1\Im}). \quad (3.37)$$

Second, let  $v$  be an arbitrary  $n$ -column and let  $v_{n+1}$  be an arbitrary complex scalar. Then we have

$$(v^\dagger, \bar{v}_{n+1})C_{n+1}(\lambda_{n+1\Im}) \begin{pmatrix} v \\ v_{n+1} \end{pmatrix} = w^\dagger C_n(\lambda_{n+1\Im})w + |v_{n+1}|^2 c_{n+1}, \quad (3.38)$$

where in the last term (3.19) was used and we denoted

$$w = v + v_{n+1} C_n^{-1}(\lambda_{n+1\mathfrak{S}}) C_{*,n+1}(\lambda_{n+1\mathfrak{S}}).$$

Thus we see that conditions (3.12)<sub>n+1</sub> for one of the signs trivially follow from (3.37) and for the other sign are equivalent to the condition  $c_{n+1}\lambda_{n+1\mathfrak{S}} > 0$ , that in its turn by Theorem 2.1 is equivalent to the condition that  $\Delta_{n+1}(x)$  has no zeroes on the  $x$ -plane and its sign is equal to the sign of  $\lambda_{n+1\mathfrak{S}}$ . Then finally (3.13)<sub>n+1</sub> follows from (3.14)<sub>n+1</sub> and the proof of lemma is completed.

**Lemma 3.4** *Let Theorem 3.1 be valid for some  $n$  and let  $\lambda_1, \dots, \lambda_{n+1}$  obey (2.11). Then for  $g_{n+1}(x)$  defined in (2.3) we have equality (3.17)<sub>n+1</sub>.*

*Proof.* By (2.3), (3.14), and (3.21)

$$g_{n+1}(x) = \frac{1}{\det A_{n+1}(x)} \begin{vmatrix} A_n(x) & A_{*,n+1}(x) \\ \Phi(x) & \Phi(x, \lambda_{n+1}) \end{vmatrix} \quad (3.39)$$

that is nothing but (3.17)<sub>n+1</sub> expanded with respect to the last column.

**Lemma 3.5** *Let Theorem 3.1 be valid for some  $n$  and let  $\lambda_1, \dots, \lambda_{n+1}$  obey (2.11). Then for  $F_{n+1}(x)$  as defined in (2.7) we have equality (3.15)<sub>n+1</sub>.*

*Proof.* We need to calculate the integral with  $F_n$  in (2.7). Thus in order to use (3.25) for this lemma we have to choose in (3.20) and, correspondingly, in (3.23) all  $\gamma_l(k) = 0$ . Then from Lemma 3.1 we get

$$F_{n+1}(x, k) = F_n(x, k) - g_{n+1}(x) \int_{-(k_{\mathfrak{S}} + \lambda_{n+1\mathfrak{S}})\infty}^{x_1} dx'_1 \partial_{x'_1} \frac{\det A_{n+1}(x'_1, x_2, k)}{\det A_n(x'_1, x_2)}. \quad (3.40)$$

Using the property (3.36) we get by (3.23) and (3.30) that

$$\begin{aligned} \frac{\det A_{n+1}(x, k)}{\det A_n(x)} &= \beta_{n+1}(x, k) - \\ &- \sum_{l,m=1}^n \frac{c_{n+1,l} + B_{n+1,l}(x)}{1 + e^{-i\lambda_l x_1 - i\lambda_l^2 x_2}} \frac{\beta_m(x, k)}{1 + e^{i\lambda_m x_1 + i\lambda_m^2 x_2}} \left( (\overline{D}_n^{-1}(x) A_n(x) D_n^{-1}(x))^{-1} \right)_{l,m}. \end{aligned}$$

We need to consider the limit  $x_1 \rightarrow -(k_{\mathfrak{S}} + \lambda_{n+1\mathfrak{S}})\infty$  of the r.h.s. Thanks to the asymptotic behavior (3.31) the first term goes to zero. By Lemma 3.2 we know that the matrix  $(\overline{D}_n^{-1} A_n D_n^{-1})^{-1}$  has finite nonzero limit and that entries of this matrix corresponding to  $\lambda_{l\mathfrak{S}} \lambda_{m\mathfrak{S}} < 0$ , as it was mentioned for the inverse matrix in the proof of Lemma 3.2, exponentially decay as  $e^{-|\lambda_{n\mathfrak{S}} x_1|}$ . Then we can write for the asymptotics of  $\beta_m(x, k)$  that  $e^{k_{\mathfrak{S}} x_1} = e^{(k_{\mathfrak{S}} + \lambda_{n+1\mathfrak{S}}) x_1} e^{-\lambda_{n+1\mathfrak{S}} x_1}$  where the first factor decays by the sign of infinity and because of condition (2.11) the second factor is majorized by some of the decaying exponents.

Now by means of (3.15)<sub>n</sub> and (3.39) we get from (3.40)

$$F_{n+1}(x, k) = \frac{1}{\det A_n(x) \det A_{n+1}(x)} \begin{vmatrix} A_n(x) & Z(x, k) \\ \Phi(x) & z_{n+1}(x, k) \end{vmatrix}, \quad (3.41)$$



where we introduced the column  $Z = (z_1, \dots, z_n)^T$  and the entry  $z_{n+1}$  as follows:

$$Z = \beta(x, k) \det A_{n+1}(x) - A_{*,n+1}(x) \begin{vmatrix} A_n(x) & \beta(x, k) \\ A_{n+1,*}(x) & \beta_{n+1}(x, k) \end{vmatrix},$$

$$z_{n+1} = \Phi(x, k) \det A_{n+1}(x) - \Phi(x, \lambda_{n+1}) \begin{vmatrix} A_n(x) & \beta(x, k) \\ A_{n+1,*}(x) & \beta_{n+1}(x, k) \end{vmatrix}.$$

By means of (3.23) with  $\gamma(k) = 0$  this can be rewritten as

$$z_j = (A_{n+2}(x, k))_{j,n+2} \det A_{n+2}^{(n+2,n+2)}(x, k) -$$

$$-(A_{n+2}(x, k))_{j,n+1} \det A_{n+2}^{(n+2,n+1)}(x, k)$$

$$z_{n+1} = \Phi(x, k) \det A_{n+2}^{(n+2,n+2)}(x, k) - \Phi(x, \lambda_{n+1}) \det A_{n+2}^{(n+2,n+1)}(x, k).$$

We see that the determinant in (3.41) is unchanged if all  $z_1, \dots, z_{n+1}$  are replaced with

$$\tilde{z}_j = \sum_{l=1}^{n+2} (-1)^{n+l} (A_{n+2}(x, k))_{j,l} \det A_{n+2}^{(n+2,l)}(x, k), \quad k = 1, \dots, n,$$

$$\tilde{z}_{n+1} = \Phi(x, k) \det A_{n+2}^{(n+2,n+2)}(x, k) + \sum_{l=1}^{n+1} (-1)^{n+l} \Phi(x, \lambda_l) \det A_{n+2}^{(n+2,l)}(x, k),$$

as all additional terms are just columns proportional to some other columns of determinant in (3.41). On the other side we see that all  $\tilde{z}_j$  for  $j = 1, \dots, n$  are equal to determinants of the matrix  $A_{n+2}(x, k)$  with row  $n+2$  replaced with  $l$ -th row of the same matrix. So all of them are equal to zero and thus the determinant in (3.41) is equal to  $\tilde{z}_{n+1} \det A_n(x)$ . On the other side  $\tilde{z}_{n+1}$  is nothing but expansion of the determinant in (3.15) $_{n+1}$  with respect to the last row. The lemma is proved.

**Lemma 3.6** *Let Theorem 3.1 be valid for some  $n$  and let  $\lambda_1, \dots, \lambda_{n+1}$  obey (2.11). Then for  $f_{n+1}(x)$  as defined in (2.8) we have equality (3.16) $_{n+1}$ .*

*Proof.* We need to calculate the integral with  $f_n$  in (2.8). Thus we have to replace in (3.20) in the last column  $\beta(x, k)$  and  $\Phi(x, k)$  with zeros. Correspondingly, Lemma 3.1 must be used with a matrix  $A_{n+1}(x, k)$  as in (3.23) with only  $\gamma$ -terms in the last column and zero on the bottom place. Using the same consideration as in Lemma 3.5 we prove that for such matrix  $A_{n+1}(x, k)$  the limit of  $\det A_{n+1}(x, k) \det A_n^{-1}(x)$  for  $x_1 \rightarrow -\lambda_{n+1} \Im \infty$  is finite, and then the integral in (2.8) is convergent. Continuing in this way we prove (3.16) $_{n+1}$  and get relation between  $B_{n+1}(k)$  and  $\gamma_{n+1}(k)$ . We omit these details, as in what follows solutions  $f_n(x, k)$  are not used.

*Proof of the Theorem 3.1* follows by induction on the base of the lemmas proved above, if we notice that for  $n = 1$  the formulation of the theorem coincides with formulas (2.3)–(2.8) for  $n = 0$  if we take into account (2.9) and condition (2.12) for  $n = 1$ . As well Eq. (3.18) $_{n+1}$  follows from (2.5) and (3.18) $_n$  thanks to (3.14) $_{n+1}$ , i.e. thanks to Lemma 3.3. Let us emphasize that reality and regularity of potentials  $u_n$  for all  $n$  are equivalent to conditions that matrices  $C_n$  are Hermitian and obey property (3.12). These properties are independent on the original potential  $u(x)$ , so they coincide with the conditions given in [22], where nondiagonal matrix  $C_n$  was introduced first time for the case  $u(x) \equiv 0$ .

**Corollary 3.1**

$$\Phi_n(x, \lambda_m) = \sum_{l=1}^n d_{l,m} \Phi_n(x, \bar{\lambda}_l), \quad (3.42)$$

where we introduced the constants

$$d_{l,m} = 2c_{l,m} \lambda_{l\Im} \prod_{\substack{j=1 \\ j \neq l}}^n \left( \frac{\bar{\lambda}_l - \lambda_j}{\bar{\lambda}_l - \bar{\lambda}_j} \right)^{\theta(\lambda_{l\Im} \lambda_{j\Im})} \prod_{j=1}^n \left( \frac{\lambda_m - \bar{\lambda}_j}{\lambda_m - \lambda_j} \right)^{\theta(-\lambda_{m\Im} \lambda_{j\Im})}. \quad (3.43)$$

*Proof.* Taking (1.11) into account we use (3.2) and (3.6) in (3.15) in order to write

$$F_n(x, k_{\Re} - i\lambda_{l\Im} + i0) - F_n(x, k_{\Re} - i\lambda_{l\Im} - i0) = -2\pi\delta(k_{\Re} - \lambda_{l\Re})\varphi_l(x), \quad l = 1, \dots, n,$$

where in analogy with (3.17) we introduced

$$\varphi_l(x) = \frac{-1}{\det A_n(x)} \begin{vmatrix} A_n(x) & e_l \\ \Phi(x) & 0 \end{vmatrix}, \quad (3.44)$$

where  $e_l = (0, \dots, 0, 1, 0, \dots, 0)^T$  is a column with 1 on the  $l$ -th place only. Thus as we already know from Corollary 2.2  $F_n$  has poles at points  $k = \bar{\lambda}_l$  and

$$\operatorname{res}_{k=\bar{\lambda}_l} F_n(x, k) = -i\varphi_l(x).$$

Then by (2.20) and Corollary 2.3

$$\Phi_n(x, \bar{\lambda}_l) = \frac{\varphi_l(x)}{2\lambda_{l\Im}} \prod_{\substack{j=1 \\ j \neq l}}^n \left( \frac{\bar{\lambda}_l - \bar{\lambda}_j}{\bar{\lambda}_l - \lambda_j} \right)^{\theta(\lambda_{l\Im} \lambda_{j\Im})}.$$

On the other side directly by (3.15), again taking into account (2.20) and Corollary 2.3 we have

$$\Phi_n(x, \lambda_m) = \sum_{l=1}^n c_{l,m} \varphi_l(x) \prod_{j=1}^n \left( \frac{\lambda_m - \bar{\lambda}_j}{\lambda_m - \lambda_j} \right)^{\theta(-\lambda_{m\Im} \lambda_{j\Im})}$$

that proves the statement.

This corollary completes the formulation of the inverse problem for the Jost solution  $\Phi_n$  as its discontinuity at the real axis was given by Corollary 2.6.

**Corollary 3.2** *Condition (2.11) for the final formulas (3.15) and (3.18) can be omitted.*

*Proof.* Indeed, this condition was relevant only for the proof that these formulas obey recursion procedure of the Theorem 2.1. On the other side Eqs. (3.15), (3.18), and (3.12) are invariant under any permutation of  $\lambda_j$ 's obeying (2.10).

## 4 Properties of potentials.

We proved that potentials given by (3.18) are real and regular and these properties are equivalent to Hermiticity of the matrix  $C_n$  defined in (3.8) and conditions (3.12). Here we demonstrate that these potentials are of the type (1.2). So we have to study their asymptotic behavior when  $x_1$  is replaced with  $x_1 - 2\mu x_2$  and  $x_2 \rightarrow \infty$ , where  $x_1$  is fixed

and  $\mu$  is a parameter determining the direction of asymptotics on the  $x$ -plane. In the same way like in Lemma 3.2 it is easy to prove that the leading term of the potential is decaying for all  $\mu \neq \lambda_{1\Re}, \dots, \lambda_{n\Re}$ . Taking into account that the original potential is rapidly decaying we have by (3.18) for this limit

$$\lim_{x_2 \rightarrow \infty} u_n(x_1 - 2\lambda_{j\Re}x_2, x_2) = -2 \lim_{x_2 \rightarrow \infty} \partial_{x_1}^2 \log \det A_n(x_1 - 2\lambda_{j\Re}x_2, x_2). \quad (4.1)$$

In analogy with Lemma 3.2 we introduce the diagonal matrix

$$\Gamma_{l,m} = \delta_{l,m} \delta_{l,j} + \delta_{l,m} (1 - \delta_{l,j}) \left( 1 + e^{-i\lambda_l x_1 - i(\lambda_l - \lambda_{j\Re})^2 x_2 - i\lambda_{j\Im}^2 x_2} \right).$$

It is obvious that in (4.1) we can replace matrix  $A_n$  with matrix  $\bar{\Gamma}^{-1} A_n \Gamma^{-1}$  without changing the value of the limit. In what follows we denote the transformations of the matrix  $A_n$  that do not affect the limit (4.1) by sign  $\simeq$ . Let now introduce in analogy with (3.33) the  $(n-1) \times (n-1)$  matrix

$$\alpha(j, \pm) = \|\alpha_{l,m}(j, \pm)\|_{l,m=1,\dots,n, l,m \neq j}, \quad (4.2)$$

$$\begin{aligned} \alpha_{l,m}(j, \pm) = & c_{l,m} \theta(\mp \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re})) \theta(\mp \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re})) + \\ & + \frac{\theta(\pm \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re})) \theta(\pm \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re}))}{i(\bar{\lambda}_l - \lambda_m)}. \end{aligned} \quad (4.3)$$

This matrix has block structure and in analogy with (3.10)–(3.11) we introduce

$$C(j, x_2) = \|\{c_{l,m} \mid \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re})x_2 > 0, \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re})x_2 > 0\}\|, \quad (4.4)$$

$$\begin{aligned} \Lambda(j, x_2) = & \|\{-i(\bar{\lambda}_l - \lambda_m)^{-1} \mid \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re})x_2 > 0, \\ & \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re})x_2 > 0\}\|, \end{aligned} \quad (4.5)$$

$$l, m = 1, \dots, n, \quad l, m \neq j,$$

and again we put the determinant of a matrix that has no entries equal to 1. Then we get

$$\det \alpha(j, \pm) = \det C(j, \mp) \det \Lambda(j, \pm).$$

For  $\Lambda(j, \pm)$  we have explicitly

$$\begin{aligned} \det \Lambda(j, \pm) = & \prod_{\substack{l=1 \\ l \neq j}}^n (2\lambda_{l\Im})^{-\theta(\pm \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re}))} \times \\ & \times \prod_{\substack{l,m=1 \\ l \neq m, l, m \neq j}}^n \left| \frac{\lambda_l - \lambda_m}{\bar{\lambda}_l - \lambda_m} \right|^{\theta(\pm \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re})) \theta(\pm \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re}))} \end{aligned} \quad (4.6)$$

and thanks to the condition (3.12) of regularity  $\det \alpha(j, \pm) \neq 0$ . Dividing  $\det(\bar{\Gamma}^{-1} A_n \Gamma^{-1})$  by  $\det \alpha$  and using the analog of Eq. (3.36) we get thanks to the asymptotic behavior (3.31) that

$$\begin{aligned} \lim_{x_2 \rightarrow \pm\infty} \det A_n(x_1 - 2\lambda_{j\Re}x_2, x_2) \simeq & c_{j,j} + \frac{e^{2\lambda_{j\Im}x_1}}{2\lambda_{j\Im}} - \\ & - \sum_{\substack{l,m=1 \\ l,m \neq j}}^n \left[ c_{j,l} \theta(\mp \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re})) + \frac{e^{i\bar{\lambda}_j x_1}}{i(\bar{\lambda}_j - \lambda_l)} \theta(\pm \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re})) \right] \times \\ & \times (\alpha(j, \pm)^{-1})_{l,m} \left[ c_{m,j} \theta(\mp \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re})) + \frac{e^{-i\lambda_j x_1}}{i(\bar{\lambda}_m - \lambda_j)} \theta(\pm \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re})) \right]. \end{aligned}$$

Thanks to (4.2) and (4.3) the elements of the matrix  $(\alpha^{-1})_{l,m}$  are proportional to corresponding  $\theta$ -functions. Then taking (4.4) and (4.5) into account we can write

$$\begin{aligned} \lim_{x_2 \rightarrow \pm\infty} \det A_n(x_1 - 2\lambda_{j\Re}x_2, x_2) &\simeq c_{j,j} - \\ &- \sum_{\substack{l,m=1 \\ l,m \neq j}}^n c_{j,l} \theta(\mp \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re}))(C(j, \mp)^{-1})_{l,m} c_{m,j} \theta(\mp \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re})) + \\ &+ e^{2\lambda_{j\Im}x_1} \left[ \frac{1}{2\lambda_{j\Im}} - \sum_{\substack{l,m=1 \\ l,m \neq j}}^n \frac{\theta(\pm \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re}))}{i(\bar{\lambda}_j - \lambda_l)} (\Lambda(j, \pm)^{-1})_{l,m} \times \right. \\ &\left. \times \frac{\theta(\pm \lambda_{m\Im}(\lambda_{m\Re} - \lambda_{j\Re}))}{i(\bar{\lambda}_m - \lambda_j)} \right]. \end{aligned}$$

Let us introduce now the matrix  $\widehat{C}(j, \pm)$  constructed by removing from the matrix  $C_n = \|c_{l,m}\|$  all rows and columns with numbers  $l \neq j$  such that  $\pm \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re}) < 0$ . Let the matrix  $\widehat{\Lambda}(j, \pm)$  be the same as  $\widehat{C}(j, \pm)$  with  $c_{l,m}$  substituted by  $-i(\bar{\lambda}_l - \lambda_m)^{-1}$ . Then again by (3.36) we get

$$\det A_n(x_1 - 2\lambda_{j\Re}x_2, x_2) \simeq \frac{\det \widehat{C}(j, \mp)}{\det C(j, \mp)} + e^{2\lambda_{j\Im}x_1} \frac{\det \widehat{\Lambda}(j, \pm)}{\det \Lambda(j, \pm)}.$$

By (4.6)

$$\frac{\det \widehat{\Lambda}(j, \pm)}{\det \Lambda(j, \pm)} = \frac{1}{2\lambda_{j\Im}} \prod_{\substack{l=1 \\ l \neq j}}^n \left| \frac{\lambda_l - \lambda_j}{\bar{\lambda}_l - \lambda_j} \right|^{2\theta(\pm \lambda_{l\Im}(\lambda_{l\Re} - \lambda_{j\Re}))}, \quad (4.7)$$

and using (4.1) we have finally that

$$\lim_{x_2 \rightarrow \pm\infty} u_n(x_1 - 2\lambda_{j\Re}x_2, x_2) = -\frac{2\lambda_{j\Im}^2}{\cosh^2(\lambda_{j\Im}x_1 + \varepsilon_{j,\pm})}, \quad (4.8)$$

where

$$e^{2\varepsilon_{j,\pm}} = \frac{\det C(j, \mp) \det \widehat{\Lambda}(j, \pm)}{\det \widehat{C}(j, \mp) \det \Lambda(j, \pm)}.$$

This proves that the potentials constructed by means of the binary Bäcklund transformations indeed give nontrivial examples of the class (1.2) since they are not decaying along directions  $x_1 + 2\lambda_{j\Re}x_2 = \text{const}$ . The two rays belonging to each direction are mutually shifted by

$$\begin{aligned} e^{2(\varepsilon_{j,+} - \varepsilon_{j,-})} &= \frac{\det \widehat{C}(j, +) \det C(j, -)}{\det \widehat{C}(j, -) \det C(j, +)} \times \\ &\times \prod_{\substack{l=1 \\ l \neq j}}^n \left| \frac{|\lambda_{j\Re} - \lambda_{l\Re}| - i(\lambda_{j\Im} \operatorname{sgn}(\lambda_{l\Re} - \lambda_{j\Re}) - |\lambda_{l\Im}|)}{|\lambda_{j\Re} - \lambda_{l\Re}| - i(\lambda_{j\Im} \operatorname{sgn}(\lambda_{l\Re} - \lambda_{j\Re}) + |\lambda_{l\Im}|)} \right|^2, \end{aligned} \quad (4.9)$$

where we used (4.7).

All consideration here was done under assumption that all  $\lambda_j$ 's are in a generic situation, in particular, that their real parts are all different. In the situation when, say,

$\lambda_{j\Re} = \lambda_{m\Re}$  the limit in (4.1) does not exist (for a generic matrix  $C_n$ ) as at large  $x_2$  there are oscillating terms. So in this case the potential does not belong to the class (1.2).

Here only the leading asymptotic behavior of the potentials  $u_n(x)$  were considered. More detailed investigation performed in [5] for the case  $n = 1$  shows that even in that simple situation the asymptotic behavior gets some rational corrections in the higher terms. Moreover, these corrections depend on the sign of  $\lambda_{1\Im}$  if the original potential  $u(x)$  is a nontrivial one. This dependence on signs of the imaginary parts of  $\lambda_j$ 's is also obvious here. Indeed, formulation of the inverse problem in (2.37) involves spectral data  $\mathcal{F}_n(k, p)$  that include only  $\lambda_j$ 's with positive imaginary parts. Such dependence of the potential on these signs is a specific two dimensional feature and as well as nondiagonal matrix in (3.42) it has no analog in the one dimensional case. We plan to discuss these aspects in more detail in a forthcoming publication.

## References

- [1] M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Theor. Math. Phys.* **93** (1992) 1200.
- [2] M. Boiti, F. Pempinelli, and A. Pogrebkov, *Theor. Math. Phys.* **99** (1994) 511.
- [3] M. Boiti, F. Pempinelli, and A. Pogrebkov, in *Nonlinear Physics. Theory and Experiment*, eds. E. Alfinito, M. Boiti, L. Martina, and F. Pempinelli, World Scientific Pub. Co., Singapore (1996), pp. 37-52.
- [4] M. Boiti, F. Pempinelli, and A. Pogrebkov, *Inverse Problems* **13** (1997) L7.
- [5] M. Boiti, F. Pempinelli, A. Pogrebkov, and B. Prinari, *Theor. Math. Phys.* **116** (1998) 741
- [6] V. S. Dryuma, *Sov. Phys. J. Exp. Theor. Phys. Lett.* **19** (1974) 381.
- [7] V. E. Zakharov and A. B. Shabat, *Funct. Anal. Appl.* **8** (1974) 226.
- [8] B. B. Kadomtsev and V. I. Petviashvili, *Sov. Phys. Doklady* **192** (1970) 539.
- [9] M. Boiti, J. Léon, L. Martina, and F. Pempinelli *Physics Letters* **A132** (1988) 432.
- [10] M. Boiti, L. Martina and F. Pempinelli, *Chaos, Solitons and fractals* **5** (1995) 2377.
- [11] V. E. Zakharov, S. V. Manakov, *Sov. Sci. Rev. – Phys. Rev.* **1** (1979) 133.
- [12] S. V. Manakov, *Physica* **D3** (1981) 420.
- [13] A. S. Fokas and M. J. Ablowitz, *Stud. Appl. Math.* **69** (1983) 211.
- [14] M. Boiti, J. Léon, and F. Pempinelli, *Phys. Lett.* **A 141** (1989) 96.
- [15] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Lecture Notes Series **49** (1991), University of Cambridge, Cambridge.
- [16] M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Inverse problems* **8** (1992) 331.
- [17] M. Boiti, F. Pempinelli, and A. Pogrebkov, *Inverse Problems* **10** (1994) 505.

- [18] M. Boiti, F. Pempinelli, and A. Pogrebkov, *Journ. Math. Phys.* **35** (1994) 4683.
- [19] Xin Zhou, *Commun. Math. Phys.* **128** (1990) 551.
- [20] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons. The Method of Inverse Scattering* Plenum, New York, 1984.
- [21] S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev *Phys. Rev. Lett.* **A 63** (1977) 205.
- [22] B. A. Dubrovin, T. M. Malanyuk, I. M. Krichever, and V. G. Makhankov *Sov. J. Part. Nucl.* **19** (1988) 252.
- [23] A. S. Fokas and A. K. Pogrebkov, unpublished.
- [24] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin 1991).
- [25] M. A. Salle, PhD Thesis, Leningrad.
- [26] M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, *Inverse problems* **7** (1991) 43.